$SU(2)$ gate composition

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1 Introduction

we can represent each $SU(2)$ elements in terms of [**?**]

$$u(\theta, \hat{\mathbf{n}}) = e^{i\theta\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\sigma}}} \tag{1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore $\hat{\mathbf{n}}$ is a 3-D unit vector.

the pair of $(\theta, \hat{\mathbf{n}})$ can be a vector in 3-D space with magnitude of θ (which is bounded) and orientation of $\hat{\mathbf{n}}$.[1]

2 Description

2.1 extraction of vector representation

for an arbitrary unitary matrix u we have

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

on the other hand

$$u = e^{i\theta\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\sigma}}} = I\cos\theta + i\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\sigma}}\sin\theta$$

where I is a 2×2 identity matrix

[1]A $SU(2)$ matrix can be represented with multiple $(\theta, \hat{\mathbf{n}})$ pairs, but one and only one of them satifies $0 \leq \theta \leq \pi$.

$$u = e^{i\theta\hat{\mathbf{n}}\cdot\vec{\boldsymbol{\sigma}}} = I\cos\theta + in_1\sigma_1\sin\theta + in_2\sigma_2\sin\theta + in_3\sigma_3\sin\theta$$

$$u = \begin{pmatrix}\cos\theta & 0\\ 0 & \cos\theta\end{pmatrix} + \begin{pmatrix}0 & in_1\sin\theta\\ in_1\sin\theta & 0\end{pmatrix} +$$

$$\begin{pmatrix}0 & n_2\sin\theta\\ -n_2\sin\theta & 0\end{pmatrix} + \begin{pmatrix}in_3\sin\theta & 0\\ 0 & -in_3\sin\theta\end{pmatrix}$$

$$u = \begin{pmatrix}\cos\theta + in_3\sin\theta & n_2\sin\theta + in_1\sin\theta\\ -n_2\sin\theta + in_1\sin\theta & \cos\theta - in_3\sin\theta\end{pmatrix}$$

now, formula below shows how to extract $(\theta, \hat{\mathbf{n}})$ from matrix elements

$$\begin{cases}\theta = \arccos(a+d)\\ n_1 = \frac{b+c}{2i\sin\theta}\\ n_2 = \frac{b-c}{2\sin\theta}\\ n_3 = \frac{a-d}{2i\sin\theta}\end{cases} \tag{2}$$

2.2 state visualization

assuming two unitary matrices

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \text{ a.k.a. Hadamard}, u_2 = \begin{pmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{4}}\end{pmatrix} \text{ a.k.a. T-Gate}$$

by ignoring their phase, we have

$$\hat{u_i} = \frac{1}{\sqrt{\det u_i}}u_i, \hat{u_i} \in SU(2)$$

we define

$$S_1 = \{\hat{u_1}, \hat{u_2}\}, S_n = \{xy | x \in S_{n-1}, y \in S_1\}$$

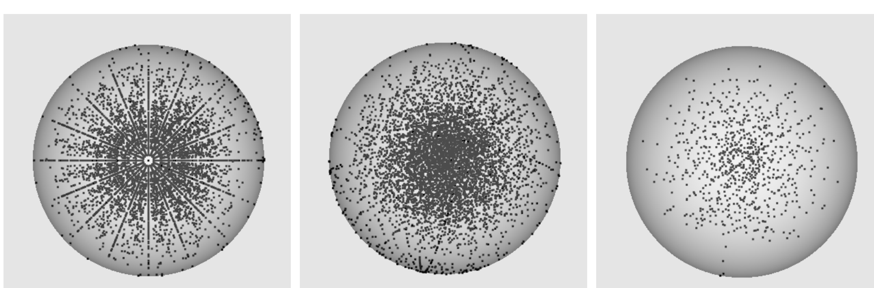

Figure 1: from left to right: $\bigcap_{i=0}^{19} S_i$,$\bigcap_{i=0}^{19} S_i$ rotated, $\bigcap_{i=0}^{12} S_i$

2.3 gate errors in $SU(2)$

as we defined error for an operator e as an estimation of u

$$\varepsilon(e,u) = \max_{|v\rangle} ||(e-u)|v\rangle||$$

then

$$\varepsilon = \max_{|v\rangle} \langle v|(e-u)^\dagger(e-u)|v\rangle$$

$$\varepsilon = 2 - \min_{|v\rangle} \langle v|(e^\dagger u + u^\dagger e)|v\rangle$$

now by replacing each $SU(2)$ matrix we have

$$\varepsilon = 2 - \min_{|v\rangle} \langle v| \Big((I\cos\theta_e + i\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}}\sin\theta_e)^\dagger(I\cos\theta_u + i\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}}\sin\theta_u) - (I\cos\theta_u + i\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}}\sin\theta_u)^\dagger(I\cos\theta_e + i\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}}\sin\theta_e)\Big)|v\rangle$$

knowing $\sigma_i = \sigma_i^\dagger$

$$\varepsilon = 2 - \min_{|v\rangle} \langle v| \Big(2I\cos\theta_e\cos\theta_u + \big((\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}}) + (\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})\big)\sin\theta_u\sin\theta_e + i(\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})\cos\theta_e\sin\theta_u - i(\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}})\cos\theta_u\sin\theta_e + i(\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}})\cos\theta_u\sin\theta_e - i(\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})\cos\theta_e\sin\theta_u\Big)|v\rangle$$

$$\varepsilon = 2 - \min_{|v\rangle} \langle v| \Big(2I\cos\theta_e\cos\theta_u + \big((\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}}) + (\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})\big)\sin\theta_u\sin\theta_e\Big)|v\rangle$$

$$\varepsilon = 2 - 2\cos\theta_e\cos\theta_u - \sin\theta_u\sin\theta_e \min_{|v\rangle} \langle v| \Big((\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}}) + (\hat{\mathbf{n}}_e\cdot\vec{\boldsymbol{\sigma}})(\hat{\mathbf{n}}_u\cdot\vec{\boldsymbol{\sigma}})\Big)|v\rangle$$

$$\varepsilon = 2 - 2\cos\theta_e\cos\theta_u - \sin\theta_u\sin\theta_e \min_{|v\rangle} \langle v| \Big((\sum_i n_{Ui}\sigma_i)(\sum_i n_{Ei}\sigma_i) + (\sum_i n_{Ei}\sigma_i)(\sum_i n_{Ui}\sigma_i)\Big)|v\rangle$$

$$\varepsilon = 2 - 2\cos\theta_e\cos\theta_u - \sin\theta_u\sin\theta_e \min_{|v\rangle} \langle v| \Big(2\sum_i n_{Ui}n_{Ei}I + i\sum_{i,j,k} n_{Ui}n_{Ej}\epsilon_{ijk}\sigma_k - i\sum_{i,j,k} n_{Ei}n_{Uj}\epsilon_{ijk}\sigma_k\Big)|v\rangle$$

finally,

$$\varepsilon = 2 - 2\cos\theta_e\cos\theta_u - 2\sin\theta_u\sin\theta_e(\hat{\mathbf{n}}_u\cdot\hat{\mathbf{n}}_e) \tag{3}$$

2.4 Solovay-Kitaev theorem for $SU(2)$

for an arbitrary unitary matrix u, minimum error of approximation with n gates will be

$$\varepsilon^*(e,n) = \min_{e\in S_n} \varepsilon(e,u)$$

in the way of proofing Solovay-Kitaev theorem is shown that [?]

$$\text{if } \varepsilon^*(e,n_0) \leq \varepsilon_0 \rightarrow \exists s,\ \forall k:\ \varepsilon^*(e,\sum_{m=0}^{k} 5^m n_0) \leq \frac{(s\varepsilon_0)^{(1.5^k)}}{s}$$

$$\text{if } \varepsilon^*(e,n_0) \leq \varepsilon_0 \rightarrow \exists s,\ \forall k:\ \varepsilon^*(e,\frac{5}{4}5^k n_0) \leq \frac{(s\varepsilon_0)^{(1.5^k)}}{s}$$

$$\text{if } \varepsilon^*(e,n_0) \leq \varepsilon_0 \rightarrow \exists s, \varepsilon^*(e,\alpha n_0) \leq \frac{(s\varepsilon_0)^{(\alpha^{\log 1.5/\log 5})}}{s}$$

$$\text{if } \log\varepsilon^*(e,n_0) \leq \delta_0 \rightarrow \exists s, \log\varepsilon^*(e,\alpha n_0) \leq (\alpha^{\log 1.5/\log 5})(\log s + \delta_0) - \log s$$

finally we can roughly write this, for a range of big α values[2]

$$\log\varepsilon^*(e,\alpha n_0) \leq \alpha^{0.252} c_1 + c_2 \quad | \ c_1 < 0 \tag{4}$$

2.5 Simulation of error

the figure below shows $\log\varepsilon^*(e,\alpha n)$ of 1000 random matrices for $1 \leq n \leq 20$

[2] other proves in other literatures [?] have reported $p = \lim_{\delta\to 0} \frac{1}{3} - \delta$ and $p = \lim_{\delta\to 0} \frac{1}{2} - \delta$ for

$$\log\varepsilon^*(e,\alpha n_0) \leq \alpha^p c_1 + c_2 \quad | \ c_1 < 0$$

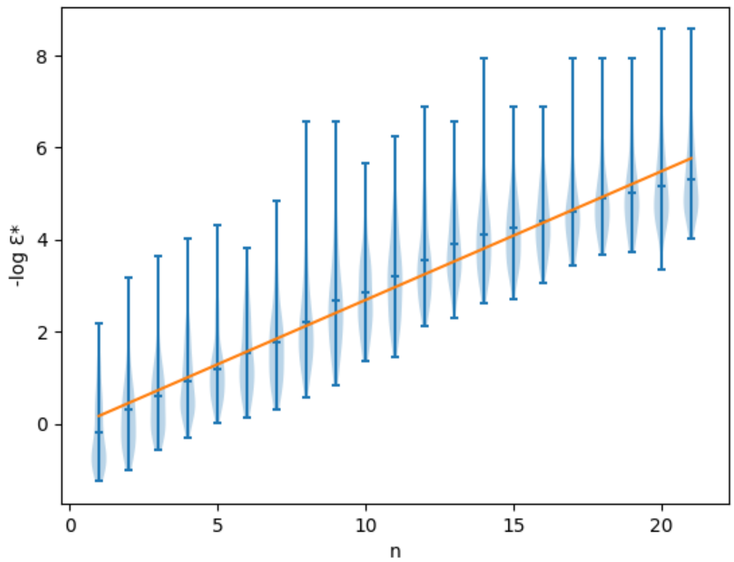

Figure 2: violin plot of $\log \varepsilon^*(e, \alpha n)$ of different samples (e) per n

trend of plot shows us that the logarithmic error changes linearly by changing n

which can be interpreted as

$$\log \varepsilon^*(e, \alpha n_0) \approx \alpha c_1 + c_2 \quad | \; c_1 < 0 \tag{5}$$

although this simulation doesn't exactly show the existence of an stronger condition for error convergence, (as it's for a little range of small ns) it may be a clue for that.

UPDATE: It's proved that for an specific set of gates, the logarithmic error is a linear function of n [**?**].

References

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[2] Horn, R. A., & Johnson, C. R. (2012). Matrix analysis. Cambridge university press.

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[5] Harrow, A. W., Recht, B., Chuang, I. L. (2002). Efficient discrete approximations of quantum gates. Journal of Mathematical Physics, 43(9), 4445-4451.