A Review of the Berry phase, Holonomy and Aharonov-Bohm Effect

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Abstract

This is a review of the Berry phase and Aharonov-Bohm effect from a purely mathematical point of view. It is known from the adiabatic theorem that the quantum state will remain at the ground state, but it can vary in phase. the Berry phase is about the geometric part of the phase gained under a cyclic evolution in the control space of Hamiltonian. This phase depends only on the path (and not the traversing velocity). Moreover, it can be written as the integral over the surface enclosed by the path with some considerations. It even goes further and can be seen as a holonomy on a bundle. Then Aharonov-Bohm effect can be seen through the eyes of differential geometry. The facts of the physical existence of potential in the absence of field (with considerations of gauge invariance) will become a simple example of the Ambrose-Singer theorem.

1 the Berry phase

Assume a Hamiltonian, parameterized with control parameters. The control space can be seen as a smooth manifold R. We use the function $p(t) : [0, T] \to P$ to traverse the path P on the manifold R over time.

Then the Schrödinger equation will be

$$i\hbar \frac{\partial \left| \psi(t) \right\rangle}{\partial t} = H(p(t)) \left| \psi(t) \right\rangle$$

Assuming starting with a ground state, and without degeneracy and closure of gap (will be discussed in sec. 1.1) and slow-changing control parameters, we can use the adiabatic theorem [1] and obtain

$$\left| \psi(t) \right\rangle = e^{i\phi(t)} \left| n_0^{H(p(t))} \right\rangle$$

[1] Although there are more assertive statements of the quantum adiabatic theorem [13, 6, 3], we will stick to a simple one by Kato [8], which can be stated as follow.

Theorem 1. *If we have a time-dependent Hamiltonian $H(t)$ with descrete eigenvalues that has no degeneracy and crossing, starting by the ground state $\left| n_0 \right\rangle$, the final state at time T, is still the ground state with the probabilty $1 - O(\frac{1}{T})$.*

where $\left|n_0^A\right\rangle$ is the ground state of A.

And then the evolution will be simplified to

$$i\hbar\dot{\phi}(t) + \left\langle n_0^{H(p(t))}\right| \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle \cdot \dot{p}(t) = E_0(p(t))$$

We already know that the Hamiltonian can be subtracted by a time-dependent identity coefficient, like $E_0(p(t))$. We call this process omitting the dynamical phase.

$$\dot{\phi}(t) = \frac{i}{\hbar}\left\langle n_0^{H(p(t))}\right| \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle \cdot \dot{p}(t)$$

$$\Delta\phi = \frac{i}{\hbar}\int_P \left\langle n_0^{H(p(t))}\right| \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle \cdot \mathrm{d}l \tag{1}$$

We can see that the phase shift is independent of traversing velocity.

Moreover, cyclic paths are more of our interests, so, we can use a generalized Stokes theorem [2] to derive

$$\begin{aligned}\Delta\phi &= \frac{i}{\hbar}\oint_P \left\langle n_0^{H(p(t))}\right| \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle \cdot \mathrm{d}l \\ &= \frac{i}{\hbar}\int_{\text{inside}(P)} \left(\boldsymbol{\nabla}_R \times \left\langle n_0^{H(p(t))}\right| \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle\right) \cdot \mathrm{d}S \\ &= \frac{i}{\hbar}\int_{\text{inside}(P)} \left(\boldsymbol{\nabla}_R \left\langle n_0^{H(p(t))}\right| \times \boldsymbol{\nabla}_R \left|n_0^{H(p(t))}\right\rangle\right) \cdot \mathrm{d}S \\ &= \frac{i}{\hbar}\int_{\text{inside}(P)} \sum_{i>0} \frac{(\langle n_0| \boldsymbol{\nabla}_R H |n_i\rangle \times \langle n_i| \boldsymbol{\nabla}_R H |n_0\rangle)}{E_i^2} \cdot \mathrm{d}S\end{aligned}$$

Which is the main result of Berry's paper [4].

Theorem 2. *the Berry phase (or the geometric phase) of a ground state (or even an excited state) under a cyclic evolution P, can be written as* [3]

$$\Delta\phi_P = \frac{i}{\hbar}\int_{inside(P)} V \cdot \mathrm{d}S \tag{2}$$

$$V := \sum_{i>0} \frac{(\langle n_0| \boldsymbol{\nabla}_R H |n_i\rangle \times \langle n_i| \boldsymbol{\nabla}_R H |n_0\rangle)}{E_i^2} \tag{3}$$

There are a few noteworthy points at this stage. One can redefine eigenvalues up to a phase factor, and this is equivalent to applying a gauge transformation like

$$\left|n_0^{\tilde{H}(p)}\right\rangle = e^{i\beta(p)}\left|n_0^{H(p)}\right\rangle$$

[2]Here we are using identities that are well-known for three-dimensional vector calculus, but they can be extended to higher dimensions. See the original paper [4, sec. 2] for more details

[3]It is also necessary to prove $\boldsymbol{\nabla} \cdot V = 0$, which is shown in [4, app. A]

which results in a change by $\beta(p(T)) - \beta(p(0))$ phase shift, which is equal to zero, for a closed path.

This was a quick review of the Berry phase definition. The motiviation behind such definitions can be formalizing quantum control theory or just understand the geometric structure of phases through adiabatic processes.

1.1 Degenerate Case

Despite that calculation of the Berry phase in a degenerate case, must be hard, but yet it has a great importance.

Intuitively from the formula 3, the vector field V will have singularity in the degenerate point and yet it's not odd that integral over the surface of a singular vector field, may results in some constant, if singularity lays inside, and 0 if singularity lays outside.

A 2-degenerate case has been studied rigorously in [4] and 3-degenerate case can be found in [7].

2 Holonomy

From a purely mathematical point of view, We should start with a principal fiber bundle B, which is made by Lie group G on the manifold M. [4]

The manifold M here acts like the control space R and the lie group G is the result of control, which is in Berry's case, the phase of the quantum state, or mathematically $U(1)$ group.

Another thing we need is a constraint to connect the manifold to the Lie group, likewise what a Hamiltonian do by connecting the control space to the phase. We use this by the concept of connection, which can be defined in a few ways, but here we define it as follow. [12]

Definition 1. *If we ignore constraints and conditions, a principal G-connection ω on B is a linear function $\mathcal{T}M \to \mathfrak{g}$, defined on each point of B.*

Note that $\mathcal{T}$ means tangent bundle and $\mathfrak{g}$ implies the Lie algebra associated with the Lie group.

This definitions shows that for an infinitesimal change in control space, how would the Lie group change respectively.

[4] Here we cannot state an exact definition of principal fiber bundle or any other necessary concept from differential geometry, and for those, we refer to [10] and [9, vol. 1, chap. I, sec. 5], but just as an intuition for unfamiliar readers, here we informally define them.

A principal fiber bundle is (more than) a manifold made by the Cartesian product of a Lie group (this is why it is called principal) and a base manifold. However, the result of the product can have a different global topological structure compared to its components. Alternatively, in other words, the neighborhood of fiber does not reflect the neighborhood of the base manifold. As an example, assuming a line fiber ($[-1, 1]$) and a closed line manifold (simply a circle), simple strip and Mobius strip are both fiber bundles (not principal, because a line fiber is not a lie group).

Definition 2. *Holonomy group of connection ω at each point of p in B can be defined as*

$$\mathrm{Hol}_p(\omega) := \{g | g \in G, \textit{a path from } p \textit{ to } pg \textit{ and consistent with } \omega \textit{ exists}\} \quad (4)$$

The idea behind this is that a loop $\gamma : [0,1] \to M$ and a path $p : [0,1] \to P$ exists such that

$$\begin{cases} p(0) = p \\ p(1) = pg \\ \exp\left[\int_0^t \omega_{p(t')}(\dot{\gamma}(t'))\mathrm{d}t'\right] = p(t) \end{cases} \quad (5)$$

By bringing back this concept to the Berry phase problem, $\mathrm{Hol}_{|\psi\rangle}$ is the group of possible phases for a closed path, starting from $|\psi\rangle$ and ending in $e^{\Delta\phi} |\psi\rangle$

2.1 Rolling Ball Example

At first, this formulation was developed to solve classical mechanics problems, such as the rolling ball. Assume a ball is rolling over an infinite surface, our control parameters are the position of ball within the surface and the connection here adds a constraint that the ball will not slip. Therefore in this case

$$\begin{cases} M = \mathbb{R}^2 \\ G = SO(3) \\ \omega_p(\delta x, \delta y) = \begin{pmatrix} 0 & 0 & \delta y \\ 0 & 0 & -\delta x \\ -\delta y & \delta x & 0 \end{pmatrix} \end{cases}$$

Then Hol must be the identity, or just rotations around one axis, or the whole rotations, but as we do not have any prefered axis, and we know that it is not identity, we can say that it is equal to $SO(3)$ [5]

2.2 Ambrose-Singer Theorem

Here we state one of the important theorems in Holonomy in order to use it to deeply understand Aharonov-Bohm effect.

But before stating, we should define curvature form of connection, which is defined pretty like other curvatures in the literature of differential geometry.

Definition 3. *The curvature form Ω is a bilinear function $\mathcal{T}M \otimes \mathcal{T}M \to \mathfrak{g}$ which is defined as curl [5] of ω with respect to M.*

Theorem 3. *For a principal fiber bundle P with a connection ω, the Lie algebra of $\mathrm{Hol}_p(\omega)$ will be generated by elements of Ω_q at all $q \in P$s that are accessible for p through the connection. [2][12]*

[5] In a more formal way, we should use extrior covariant derivative together with defining both ω and ω over tangent bundle of P instead of M. But in our case and our needs, this is just enough. [9]

We leave any further discussion on this theorem here and we will continue in the next section, after a recap of Aharonov-Bohm effect.

3 Aharonov-Bohm Effect

Returning back to the Berry phase, Here we state a problem. If we have a charged particle under time-independent electromagnetic potentials, with fully controlled path in the space, which means that $R = \mathbb{R}^3$, then we are looking for the phase it gains while moving in a closed path.

This problem can be solved using 2. And for an special case that electric potential is zero, as it is solved in [4] and the result is

$$\Delta\phi_P = \frac{q}{\hbar}\int_{\text{inside}(P)} A \cdot \mathrm{d}S \tag{6}$$

Nevertheless, to understand why this result is so important, we need to describe the physics behind it. Assume a typical double-slit experiment with a selenoid between two slits, as in fig. 3. The amazing part of the result is that even though both paths are going through a space where $B = 0$ and they must not be aware of the solenoid, we see a phase shift caused by the solenoid. Moreover, this experiment can answer the question of which comes first, the potential or the field, which is seemingly to be the potential. Also, it must be mentioned that this experiment is compatible with the additional degree of freedom in the potentials. [1]

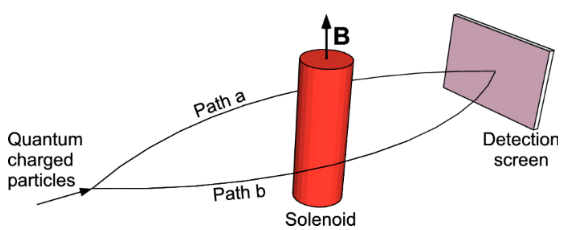

Figure 1: Aharonov-Bohm setup, from [11]

Finally, we want to study this effect using the Ambrose-Singer theorem. The set of Hol_p can specify which phase-shifts are possible to see through an interferometer in an experimental setup. A trivial Hol_p (a group of identity) means that interferometer cannot measure any phase-shift. Then here we try to calculate Hol_p for a point in space. At the point we have $B = 0$ and even maybe $A = 0$, but the Hol depends on the curvature. We first calculate the connection itself and then the curvature.

Assume an infinitesimal change in position δr, then the phase-shift will be $e^{\frac{iq}{\hbar}A\cdot\delta r}$, then the $\omega = \frac{iq}{\hbar}\begin{pmatrix}A_x & A_y & A_z\end{pmatrix}$, where i is the only generator of $U(1)$.

As a result $\Omega = \frac{iq}{\hbar}\nabla \times A = \frac{iq}{\hbar}B$ which means that $\mathrm{Hol}_p = U(1)$ if and only if there exists some q where $B \neq 0$ and q is accessible for p through a controllable path, which is true in our case, and therefore we can see an interference in this experiment.

4 Final Word

This review was an attempt to see this effect from an uncommon mathematical point of view. I did not try to cover all of the mathematical results beyond the Berry phase, as I obviously missed the Chern numbers in that case.

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